

MATH1010 Revision Exercise 2 (with some corrections)

1. Compute the first derivative of each of the functions below:

- (a)  $\arctan(x+1)$     (b)  $\arcsin(x^2)$     (c)  $x \arcsin(2x)$     (d)  $(\arcsin(x))^2$   
 (e)  $\frac{\arctan(x)}{x}$     (f)  $\sqrt{\arctan(2x)}$     (g)  $\arctan(\ln(x))$     (h)  $\arcsin(\sqrt{1-x^2})$

2. For each of the relations below, find  $\frac{dy}{dx}$  for the function  $y$  implicitly defined by the relation:

- (a)  $x = 4y - y^3$     (b)  $x = y - \frac{1}{y}$     (c)  $x = (3y+2)^{10}$     (d)  $x = (4-y)(3+y^2)$   
 (e)  $x = y^{-2} \sin(y)$     (f)  $x = \sqrt{\frac{y+1}{y+2}}$     (g)  $x^2 + y^2 = 4$     (h)  $x^2 + y^2 - 3x + 1 = 0$   
 (i)  $4y^2 + xy - 6x^2 = 0$     (j)  $2x^3 + y^3 - 3x^2y = 1$     (k)  $x^2 \sin(y) - y \cos(x) = 2$   
 (l)  $x \cos(y) + y^2 \sin(x) = 0$

3. Let  $n$  be a positive integer. Let  $f(x) = (1-x^2)^n$  for any  $x \in \mathbb{R}$ .

- (a) Show that  $(1-x^2)f'(x) + 2nxf(x) = 0$  for any  $x \in \mathbb{R}$ .  
 (b) Show that  $(1-x^2)f^{(n+2)}(x) - \cancel{(2n+1)}x f^{(n+1)}(x) + \cancel{n(n+1)}f^{(n)}(x) = 0$  for any  $x \in \mathbb{R}$ .  
 $2$   $-(n-2)(n+1)$

4. Let  $f(x) = e^x \ln(1+x)$  for any  $x \in (-1, +\infty)$ .

- (a) Show that  $(1+x)f''(x) - \cancel{(1+2x)}f'(x) + xf(x) = 0$  for any  $x \in (-1, +\infty)$ .  
 (b) Let  $n$  be a non-negative integer. Show that  $(1+x)f^{(n+3)}(x) + (n-2x)f^{(n+2)}(x) + (x-2n-2)f^{(n+1)}(x) + (n+1)f^{(n)}(x) = 0$  for any  $x \in \mathbb{R}$ .  
 $(n+1)$

5. Let  $f(x) = \frac{\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}}$  for any  $x \in \mathbb{R}$ .

- (a) Show that  $(1+x^2)f'(x) + xf(x) = 1$  for any  $x \in \mathbb{R}$ .  
 (b) Let  $n$  be a non-negative integer. Show that  $(1+x^2)f^{(n+2)}(x) + (2n+3)xf^{(n+1)}(x) + (n+1)^2 f^{(n)}(x) = 0$  for any  $x \in \mathbb{R}$ .  
 $2 \arcsin x$   
 $\sqrt{1-x^2}$

6. Let  $f(x) = (\arcsin(x))^2$  for any  $x \in (-1, 1)$ .

- (a) Show that  $(1-x^2)f'(x) - xf(x) = 2$  for any  $x \in \mathbb{R}$ .  
 (b) Let  $n$  be a positive integer. Show that  $(1-x^2)f^{(n+2)}(x) - \cancel{(2n+1)}x f^{(n+1)}(x) - \cancel{(n)^2} f^{(n)}(x) = 0$  for any  $x \in \mathbb{R}$ .  
 $2n+3$   $(n+1)^2$

7. Let  $f : [3, 6] \rightarrow \mathbb{R}$  be a continuous function. Suppose  $f$  is differentiable on  $(3, 6)$ , and  $|f'(x) - 9| \leq 3$  on  $(3, 6)$ . Show that  $18 \leq f(6) - f(3) \leq 36$ .

8. Let  $\beta \in (1, +\infty)$ . Let  $f : (0, +\infty) \rightarrow \mathbb{R}$  be the function defined by  $f(x) = x^\beta + \beta - 1 - \beta x$  for any  $x \in (0, +\infty)$ .

- (a) i. Compute  $f'$ .  
 ii. Show that  $f$  is strictly decreasing on  $(0, 1]$ .  
 iii. Show that  $f$  is strictly increasing on  $[1, +\infty)$ .  
 iv. Determine whether  $f$  attains the maximum and/or the minimum on  $(0, +\infty)$ .  
 (b) Hence, or otherwise, show that  $(1+r)^\beta \geq 1 + \beta r$  for any  $r \in (-1, +\infty)$ .

9. Prove the following inequalities:

- (a)  $\frac{x}{1+x^2} < \arctan(x) < x$  for any  $x \in (0, +\infty)$ .

(b)  $0 < \ln(1+x) - \frac{2x}{2+x} < \frac{x^3}{12}$  for any  $x \in (0, +\infty)$ .

10. (a) Prove that  $1 - \frac{x^2}{2} < \cos(x)$  for any  $x \in (0, 2\pi]$ .

(b) Prove that  $\cos(x) < 1 - \frac{x^2}{2} + \frac{x^4}{24}$  for any  $x \in (0, 2\pi]$ .

(c) Prove that  $1 - \frac{x^2}{2} < \cos(x) < 1 - \frac{x^2}{2} + \frac{x^4}{24}$  for any  $x \in (2\pi, +\infty)$ .

(d) Prove that  $1 - \frac{x^2}{2} < \cos(x) < 1 - \frac{x^2}{2} + \frac{x^4}{24}$  for any  $x \in \mathbb{R} \setminus \{0\}$ .

11. Apply L'Hôpital's Rule to evaluate each of the limits below.

(a)  $\lim_{x \rightarrow 0} \frac{x + \tan(x)}{\sin(2x)}$  (b)  $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{\sin(x)}$  (c)  $\lim_{x \rightarrow 0} \frac{\arctan(x)}{x}$  (d)  $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{2 \ln(1+x)}$

(e)  $\lim_{x \rightarrow 0} \frac{x^2 + 3x + 4}{3x^3 + 5}$  (f)  $\lim_{x \rightarrow 0} \frac{x - \sin(x)}{2x^3}$  (g)  $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin(x)}$  (h)  $\lim_{x \rightarrow 1} \frac{e^{x-1} - x}{(x-1)^2}$

(i)  $\lim_{x \rightarrow 0} \frac{24 \cos(x) - 24 - 12x^2 + x^4}{\sin^6(x)}$  (j)  $\lim_{x \rightarrow 0} \frac{x \tan(x)}{1 - \sqrt{1-x^2}}$  (k)  $\lim_{x \rightarrow +\infty} \frac{\ln(e^x + x^2)}{x^2}$

(l)  $\lim_{x \rightarrow 1} \frac{1 + \ln(x) - x^x}{1 + \ln(x) - x}$  (m)  $\lim_{x \rightarrow 0^+} \frac{(\ln(x))^5}{\sqrt[5]{x}}$  (n)  $\lim_{x \rightarrow +\infty} \frac{\ln(1 + xe^{2x})}{\sin^2(x)}$

(m)  $\lim_{x \rightarrow 0^+} \frac{\ln(\sin(\alpha x))}{\ln(\sin(\beta x))}$ . (Here  $\alpha, \beta$  are positive real numbers.)

12. Evaluate each of the limits below. When necessary, apply L'Hôpital's Rule.

(a)  $\lim_{x \rightarrow 0^+} x^2 e^{(-x^{-2})}$  (b)  $\lim_{x \rightarrow 0^+} \sin(x) \ln(x)$  (c)  $\lim_{x \rightarrow 0^+} x \csc(2x)$  (d)  $\lim_{x \rightarrow +\infty} x \left[ \left(1 + \frac{1}{x}\right)^x - e \right]$

13. Evaluate each of the limits below. When necessary, apply L'Hôpital's Rule.

(e)  $\lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\arctan(x)} \right)$  (f)  $\lim_{x \rightarrow 0^+} \left( \csc^2(x) - \frac{1}{x^2} \right)$

(g)  $\lim_{x \rightarrow 1^+} \left( \frac{x^2}{(1-x)^2} - \frac{1}{(\ln(x))^2} \right)$  (h)  $\lim_{x \rightarrow 0^+} \left( \cot(x) - \frac{1}{x} \right)$

14. Evaluate each of the limits below. When necessary, apply L'Hôpital's Rule.

(a)  $\lim_{x \rightarrow +\infty} x^{\frac{1}{x}}$  (b)  $\lim_{x \rightarrow 0^+} x^{\sin(x)}$  (c)  $\lim_{x \rightarrow 0^+} \left( \ln \left( \frac{1}{x} \right) \right)^x$  (d)  $\lim_{x \rightarrow +\infty} \left( 1 - \frac{1}{x} \right)^{-x}$

(e)  $\lim_{x \rightarrow +\infty} \left( 1 + \frac{2}{x} \right)^{-x}$  (f)  $\lim_{x \rightarrow +\infty} \left( 1 + \frac{3}{x^2} \right)^x$  (g)  $\lim_{x \rightarrow +\infty} \left( \frac{x+1}{x-1} \right)^x$  (h)  $\lim_{x \rightarrow +\infty} \left( \frac{x^2+1}{x^2-1} \right)^{x^2}$

(i)  $\lim_{x \rightarrow +\infty} \left( \frac{x^2 - 2x - 3}{x^2 - 3x - 28} \right)^x$  (j)  $\lim_{x \rightarrow \frac{\pi}{2}^-} (\tan(x))^{\cos(x)}$  (k)  $\lim_{x \rightarrow 0^+} (1 + \sin(x))^{\frac{1}{x}}$

(l)  $\lim_{x \rightarrow 0^+} (1 + \sin^2(x))^{\frac{1}{x}}$  (m)  $\lim_{x \rightarrow 1^+} x^{\frac{e^x}{1-x}}$  (n)  $\lim_{x \rightarrow 0^+} (1 - \cos(x))^{\frac{1}{\ln(x)}}$

(o)  $\lim_{x \rightarrow \frac{\pi}{2}^-} (\cos(x))^{\ln(\sin(x))}$  (p)  $\lim_{x \rightarrow 0} \left( \frac{\arcsin(x)}{x} \right)^{\frac{1}{x^2}}$  (q)  $\lim_{x \rightarrow 0} \left( \frac{\sin(x)}{x} \right)^{\frac{1}{x^2}}$

15. Evaluate each of the limits below. Think carefully whether to apply L'Hôpital's Rule or not.

(a)  $\lim_{x \rightarrow +\infty} \frac{x + \sin(x)}{x - \sin(x)}$  (b)  $\lim_{x \rightarrow +\infty} \frac{e^x + x \sin(x) + \cos(x)}{e^x + \cos(x)}$  (c)  $\lim_{x \rightarrow +\infty} \frac{x^2 + \sin(2x)}{(2x^3 + x + \sin(x))e^{\sin(x)}}$